6. Yu. A. Boikov, B. M. Gol'tsman, and S. F. Sipenko, "Method for determining the thermal conductivity of thin films," Prib. Tekh. Eksp., No. 2, 230-232 (1975).
7. R. L. Filler and P. Lindenfeld, "Specific heat and thermal conductivity measurements on thin films with a pulse method," Rev. Sci. Instrum., 46, No. 4, 439-442 (1975).
8. O. Budke and J. Krempasky, "Measurement of thermal parameters of thin films and plane-parallel plates located on a thermally conductive base," Phys. Status Solidi, 21, 175-182 (1967).
9. F. Kelemen, "Pulse method for the measurement of the thermal conductivity of thin films," Thin Solid Films, 36, No. 1, 199-203 (1976).
10. T. C. Boyce and Y. W. Chung, "A new technique for the measurement of thermal conductivity of thin films," Thermal Conductivity, New York—London (1976), pp. 499-506.
11. L. P. Filippov, Measurement of the Thermal Properties of Solid and Liquid Metals at High Temperatures [in Russian], Moscow State Univ. (1967), p. 111.
12. A. V. Lykov, Theory of Heat Conduction [in Russian], Vysshaya Shkola, Moscow (1967), pp. 365-371.

APPLICATION OF THE VARIATIONAL PRINCIPLE TO THE SOLUTION OF GENERALIZED COUPLED PROBLEMS IN THERMOELASTICITY OF INHOMOGENEOUS MEDIA

Yu. M. Kolyano and Z. I. Shter

UDC 539.3

A variational principle is formulated for coupled thermoelasticity for inhomogeneous media. The problem of thermoelastic energy dissipation accompanying transverse oscillations of an inhomogeneous isotropic cantilevered beam is solved.

The application of direct methods to the solution of coupled problems in thermoelasticity for inhomogeneous media encounters considerable mathematical difficulties. The development of approximate methods for solving coupled problems based on variational principles is promising.

We shall formulate the variational principle for coupled thermoelasticity for inhomogeneous media. We shall examine the isothermal energy of deformation

$$W=\frac{1}{2}\int\limits_{\Omega} c_{ijkl}(x_s)\, e_{kl}e_{ij}dV, \tag{1}$$

where Ω is the volume of the body.

Let us transform (1) taking into account the Duhamel—Neumann equation for inhomogeneous media and the equations of motion. As a result, we obtain

$$\int\limits_{\Omega} X_i\delta u_i dV+\int\limits_{A} P_i\delta u_i dA-\int\limits_{\Omega}\rho(x_s)\,\ddot{u}_i\delta u_i dV=\delta W-\int\limits_{\Omega}\beta_{ij}(x_s)\, t\delta e_{ij}dV, \tag{2}$$

where A is the surface of the body.

We introduce the vector $\vec{H}$, related to the heat flux vector by the relation

$$\vec{q}=t_0\dot{\vec{H}}. \tag{3}$$

Taking into account the generalized law of heat conduction

$$lq_i=-\lambda_{ij}^t(x_s)\, t_{,j} \tag{4}$$

Khmel'nitskii Technological General Service Institute. Institute of Applied Problems in Mathematics and Mechanics, Academy of Sciences of the Ukrainian SSR, Lvov. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 42, No. 1, pp. 102-107, January, 1982. Original article submitted October 15, 1980.

 0022-0841/82/4201-0084$07.50

and relation (3). we find

$$t_{,i} = -t_0 k_{ij}(x_s)\, l H_j, \tag{5}$$

where $k_{ij}(x_s)$ is the inverse matrix for the coefficients of thermal conductivity $\lambda^t_{ij}(x_s)$; $l = [1 + \tau_r(\partial/\partial\tau)]$; τ_r is the relaxation time for the heat flux.

Multiplying (5) by δH_i, integrating over Ω, and using the equation for the rate of growth of entropy

$$t_0\dot{S} = -\operatorname{div}\vec{q} = c_e(x_s)\,\dot{t} + \beta_{ij}(x_s)\, t_0 e_{ij},$$

i.e.,

$$H_{i,i} = -\frac{c_e(x_s)}{t_0}\,\dot{t} + \beta_{ij}(x_s)\, e_{ij},$$

we obtain the equation

$$\int_A t\delta H_i n_i dA + \frac{1}{t_0}\int_\Omega c_e t\delta t dV + \int_\Omega \beta_{ij} t\delta e_{ij} dV + t_0 \int_\Omega k_{ij}\delta H_i l H_j dV = 0. \tag{6}$$

Eliminating from Eqs. (2) and (6) terms that contain variations of deformations, we arrive at a variational equation for the generalized coupled problem of thermoelasticity of anisotropic inhomogeneous bodies

$$\delta(W + P + D) = \int_\Omega (X_i - \rho(x_s)\,\ddot{u}_i)\,\delta u_i dV + \int_A P_i \delta u_i dA - \int_A t\delta H_n dA, \tag{7}$$

where

$$\delta H_n = n_i \delta H_i; \quad P = \frac{1}{2t_0}\int_\Omega c_e(x_s)\, t^2 dV;$$

$$\delta D = t_0 \int_\Omega k_{ij}(x_s)\,\delta H_i l \dot{H}_j dV.$$

Thus, variation of the sum of the work of deformation, the thermal potential, and the dissipation function equals the virtual work of external forces, inerial forces, and heating of the surface.

Let us represent the components of the displacement vector u_i and the components of the vector H_i as follows:

$$u_i = \sum_{j=1}^{n} u_{ij}(x_s)\, q_j(\tau), \quad H_i = \sum_{j=1}^{n} H_{ij}(x_s)\, q_j(\tau), \tag{8}$$

where q_j are generalized coordinates. We assume that δu_i and δH_i do not depend on time. Then, defining

$$\delta u_i = \frac{\partial u_i}{\partial q_j}\delta q_j, \quad \delta H_i = \frac{\partial H_i}{\partial q_j}\delta q_j, \quad \frac{\partial H_i}{\partial q_j} = \frac{\partial \dot{H}_i}{\partial \dot{q}_j},$$

$$\delta K = \frac{d}{d\tau}\left(\frac{\partial K}{\partial q_j}\right)\delta q_j, \quad \delta u_i = v_i d\tau, \quad \delta t = \dot{t} d\tau,$$

we shall represent (7) in the form of the Lagrangian equations of motion

$$\frac{\partial(W + P)}{\partial q_j} + \frac{\partial D_\tau}{\partial \dot{q}_j} + \frac{d}{d\tau}\left(\frac{\partial K}{\partial q_j} + \tau_r \frac{\partial D_\tau}{\partial \dot{q}_j}\right) = Q_j, \tag{9}$$

where

$$D_\tau = \frac{t_0}{2}\int_\Omega k_{ij}(x_s)\,\dot{H}^2 dV; \tag{10}$$

$$Q_j = \int_\Omega X_i \frac{\partial u_i}{\partial q_j} dV + \int_A \left(P_i \frac{\partial u_i}{\partial q_j} - t_{n_i} \frac{\partial \dot{H}_i}{\partial \dot{q}_j} \right) dA$$

is the generalized force.

We shall use the variational principle (9) to solve the problem of thermoelastic energy dissipation accompanying transverse oscillations of an inhomogeneous isotropic cantilevered beam. Assume that the beam with a rectangular transverse cross section has height h, width b, and length l. The axis of the beam is directed along the OX axis and the origin of coordinates is located at the fixed end of the beam. Based on the elementary theory of bending of a beam, we have

$$\varepsilon_x = -z \frac{d^2 u_z}{dx^2}; \quad \varepsilon_y = \varepsilon_z = -\nu(x_s)\,\varepsilon_x; \quad \varepsilon_{xy} = \varepsilon_{yz} = \varepsilon_{zx} = 0; \quad \varepsilon_{kk} = \varepsilon_x + \varepsilon_y + \varepsilon_z = -z\,[1 - 2\nu(x_s)] \frac{d^2 u_z}{dx^2}, \tag{11}$$

where $\nu(x_s)$ is the Poisson coefficient of an inhomogeneous beam; u_z is the deflection. We shall assume that the change in deflection of the beam has the following form:

$$u_z = q_1 \left(1 - \cos \frac{\pi x}{2l}\right), \tag{12}$$

where q_1 is the deflection of the free end of the beam and is a generalized coordinate. We assume that the entropy density varies similarly to the deflection, i.e.,

$$S = \frac{2 q_2 z}{h} \cos \frac{\pi x}{2l}, \tag{13}$$

where q_2 is the generalized coordinate for entropy.

Let us calculate (W + P), K, and D_T taking into account (11), (12), and (13):

$$W + P = \frac{1}{2} a_{11} q_1^2 + a_{12} q_1 q_2 + \frac{1}{2} a_{22} q_2^2, \quad K = \frac{1}{2} m_{11} \dot{q}_1^2, \quad D_T = \frac{1}{2} b_{22} \dot{q}_2^2, \tag{14}$$

where

$$m_{11} = \int_\Omega \rho(x_s) \left(1 - \cos \frac{\pi x}{2l}\right)^2 dV; \tag{15}$$

$$a_{11} = \int_\Omega z^2 [1 - 2\nu(x_s)]^2 \left(1 + \frac{\pi^2}{4l^2} \cos \frac{\pi x}{2l}\right)^2 \left[k(x_s) + \frac{t_0}{c_e(x_s)} \gamma^2(x_s)\right] dV; \tag{16}$$

$$a_{12} = \frac{4t_0}{h} \int_\Omega z^2 \gamma(x_s)\,[1 - 2\nu(x_s)] \cos \frac{\pi x}{2l} \left(1 + \frac{\pi^2}{4l^2} \cos \frac{\pi x}{2l}\right) dV; \tag{17}$$

$$a_{22} = \frac{4t_0}{h^2} \int_\Omega \frac{z^2}{c_e(x_s)} \cos^2 \frac{\pi x}{2l} dV; \tag{18}$$

$$b_{22} = \frac{t_0}{h^2} \int_\Omega \frac{1}{\lambda^t(x_s)} \left(\frac{h^2}{4} - z^2\right) \cos^2 \frac{\pi x}{2l} dV. \tag{19}$$

Substituting the values (15)-(19) into (9), taking into account the fact that $Q_j = 0$, we obtain the following system of differential equations:

$$m_{11} \ddot{q}_1 + a_{11} q_1 + a_{12} q_2 = 0, \tag{20}$$

$$\tau_r b_{22} \ddot{q}_2 + b_{22} \dot{q}_2 + a_{22} q_2 + a_{12} q_1 = 0. \tag{21}$$

We shall assume that the beam undergoes harmonic oscillations when

$$q_2 = \exp(i\omega\tau), \quad \dot{q}_2 = i\omega q_2, \quad \ddot{q}_2 = -\omega^2 q_2. \tag{22}$$

Transforming (20) and (21), taking into account (22), we obtain

$$m_{11}\ddot{q}_1 + q_1\left[a_{11} - \frac{a_{12}^2(a_{22}-\tau_r\omega^2 b_{22})}{(a_{22}-\tau_r\omega^2 b_{22})^2 + b_{22}\omega^2} + i\,\frac{a_{12}^2\, b_{22}\omega}{(a_{22}-\tau_r\omega^2 b_{22})^2 + b_{22}^2\,\omega^2}\right] = 0, \tag{23}$$

$$M = a_{11} - \frac{a_{12}^2(a_{22}-\tau_r\omega^2 b_{22})}{(a_{22}-\tau_r\omega^2 b_{22})^2 + b_{22}\omega^2} \tag{24}$$

is the modulus of rigidity of an inhomogeneous beam, while

$$F_{\text{d}} = \frac{a_{12}^2\, b_{22}\omega}{(a_{22}-\tau_r\omega^2 b_{22})^2 + b_{22}\omega^2} \tag{25}$$

is the equivalent damping force, which vanishes for $\omega = 0$ and $\omega \to \infty$. We shall represent (23) in the following form

$$m_{11}\ddot{q}_1 + q_1(M + iF_{\text{d}}) = 0. \tag{26}$$

Let us denote the damping coefficient in terms of β:

$$\beta = \sqrt{\frac{\sqrt{M^2 + F_{\text{d}}^2} + M}{2m_{11}}}, \quad \bar{\omega} = \sqrt{\frac{\sqrt{M^2 + F_{\text{d}}^2} - M}{2m_{11}}}; \tag{27}$$

then,

$$q_1 = \exp(-\beta\tau)\cos\bar{\omega}\tau. \tag{28}$$

For a frequency of oscillations $\omega = \sqrt{a_{22}/b_{22}\tau_r}$, the modulus of rigidity attains the maximum value equal to a_{11}. Let us determine β for different values of the frequency of oscillation:

$$\omega = 0 \quad \beta = \sqrt{\frac{a_{11} - \frac{a_{12}}{a_{22}}}{m_{11}}}, \quad \omega \to \infty \quad \beta = \sqrt{\frac{a_{11}}{m_{11}}},$$

while if $\omega = \omega_{cr}$, then

$$\beta = \sqrt{\frac{\sqrt{a_{11}^2 + \frac{a_{12}^4\, b_{22}\tau_r}{a_{22}}} - a_{11}}{2m_{11}}}. \tag{29}$$

For $\omega = \omega_{cr}$, the damping coefficient takes on the greatest value. Let us examine how the changes in the thermophysical characteristics of the beam affect the energy dissipation.

1. Let the thermophysical characteristics vary according to the law

$$p(y) = p_0\exp(ny), \tag{30}$$

where p_0 and n are given parameters. In this case, we obtain:

$$a_{11} = (k_0 + t_0\gamma_0^3 c_{e_0}^{-1})\,\frac{h^3\left(l + \frac{\pi^4}{32}\,l^{-4}\right)}{n}\left[(\exp(nb) - 1) - 2\nu_0(\exp(2nb) - 1) + \frac{4}{3}\,\nu_0^2(\exp(3nb) - 1)\right], \tag{31}$$

$$a_{12} = \frac{2\nu_0 t_0 h^2(4l^2 + \pi^3)}{3\pi l n}\,[\exp(nb)(1 - \nu_0\exp(nb) - 1 - \nu_0)], \tag{32}$$

$$a_{22} = 4t_0 lh(3c_{e_0} n)^{-1}(1 - \exp(-nb)), \tag{33}$$

$$m_{11} = n^{-1}\rho_0 lh(\exp(nb) - 1)\left(\frac{3}{2} + \frac{4}{\pi}\right), \tag{34}$$

$$b_{22} = t_0 lh(1 - \exp(-nb))(8\lambda_0^t n)^{-1}. \tag{35}$$

2. Let the thermophysical characteristics vary according to the law

$$p(y) = p_0 + (p_1 - p_0)\,S_-(y), \tag{36}$$

where $S\_(y)$ is the asymmetric unit function. Then

$$a_{11}=\frac{1}{3}\, lh^3b\left(1+\frac{\pi^4}{8l^3}+\frac{\pi^3}{4l^3}\right)\left\{\frac{1}{2}\,(1+4\nu_0)\left(k_1-k_0+t_0\left(\frac{1}{c_{e_1}}-\frac{1}{c_{e_0}}\right)\times\right.\right.$$

$$\left.\left.\times\left[\gamma_1^2+\frac{(\gamma_1^2-\gamma_0^2)}{c_{e_1}}\right]\right)+2\left[(\nu_1^2-\nu_0^2)\,k_1+t_0\gamma_1^2\,c_{e_1}^{-1}\right]\;+(1+4\nu_0^2)\left(k_0+\frac{t_0\gamma_0^2}{c_{e_0}}\right)\right\}, \tag{37}$$

$$a_{12}=\pi(1+\pi)\,t_0h^2b\left[1-2\nu_0\gamma_0(1+\frac{\gamma_1}{\gamma_0}+\frac{\nu_1}{\nu_0}\right)-4\nu_1\gamma_1\Big]\,(3l)^{-1}, \tag{38}$$

$$a_{22}=\frac{t_0hlb}{3}\left(\frac{1}{c_{e_0}}+\frac{1}{c_{e_1}}\right), \tag{39}$$

$$m_{11}=(-3)\,l^2hb\,\frac{(\rho_0+\rho_1)}{\pi}, \tag{40}$$

$$b_{22}=-\frac{blht_0}{24}\left(\frac{1}{2}+\frac{2}{\pi}\right)\left(\frac{1}{\lambda_1^t}+\frac{1}{\lambda_0^t}\right). \tag{41}$$

Comparison of expressions (31)-(35) and (37)-(41) shows that the nature of the inhomogeneity of the material has a large effect on the magnitude of the energy dissipated with oscillations of the beam.

Thus, it follows from what was said above that with the help of the given variational of principle, it is possible to determine the magnitude and characteristics of the thermoelastic energy dissipation with oscillations of inhomogeneous bodies.

NOTATION

x_s (s = 1, 2, 3), rectilinear Cartesian coordinates; τ, time; λ_{ij}^t (i, j = 1, 2, 3), coefficients of thermal conductivity of an anisotropic body; t_0, temperature of the body in an unstressed state; t, increase in temperature at points in the body; S, entropy; e_{ij}, components of the deformation in Cartesian numbered axes; c_{ijkl}, elastic coefficients of inhomogeneous anisotropic bodies; β_{ij}, coefficients of an inhomogeneous anisotropic body, taking into account the mechanical and thermal properties of the material; c_e, volume heat capacity at constant deformation; ρ, density of the inhomogeneous anisotropic body; X_i, components of the vector of mass forces; P_i, components of the vector of surface forces; k, l = 1, 2, 3.

LITERATURE CITED

1. Ya. S. Podstrigach and Yu. M. Kolyano, Generalized Thermomechanics [in Russian], Naukova Dumka, Kiev (1976).
2. A. D. Kovalenko, Foundations of Thermoelasticity [in Russian], Naukova Dumka, Kiev (1970).
3. Yu. M. Kolyano and Z. I. Shter, "Thermoelasticity of inhomogeneous media," Inzh.-Fiz. Zh., 38, No. 6, 1111-1114 (1980).